

Mathematics: Roots and Radicals



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1. Powers

The undoing (inverse operation) of addition is subtraction, while the undoing (inverse operation) of multiplication is division. What is the undoing of squaring? Or cubing?

Before we answer that, let's review the definition of a power.

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

2 is called the **base** and 4 is the **exponent** to which the base is raised. The entire expression 2^4 is called a **power**.

Using this definition, $3^3 = 3 \cdot 3 \cdot 3 = 27$.

Letting x be any number and n be a positive integer, we have

By substituting various values for x and n , one can create a variety of powers.

By understanding how these powers are created, you will learn how to simplify and undo powers.

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2. Area

You have probably calculated the area of a square and the volume of a cube before. What you may not have seen or noticed is that the simplest formulas for each calculation involve exponents or powers, and a simple question can lead to a difficult problem. Click the icon to see the formula for a square.

Let A = area of a square. Let s = side length.

$$A = s \times s = s^2$$

Notice the area of a square is "s squared." The designation of the term "squared" to describe powers of two originated from this formula.

Example 1: Calculate the area of a square whose sides are 4 cm.

$$A = s^2 = 4^2 = (4)(4) = 16$$

The area of the square is 16 square cm.

Example 2: Calculate the length of the side of a square whose area is 25 square cm.

$$A = s^2$$
$$25 = s^2$$

Guessing and checking with a few numbers quickly leads to the solution that $s = 5$ cm.

But what happens when we try to find the length of the side for a square whose area is 23 square cm?

The answer has to be between 4 and 5. A square of side 4 cm would have an area of 16 square cm, while a square of side 5 cm would have an area of 25 square cm. Hence, our answer must be between 4 and 5, and is probably closer to 5. Try squaring 4.8 and 4.9 on your calculator, and see how close you can get to 23. Click the icon to see the result of our guess-and-check process.

Guessing and checking will give you a reasonably accurate answer, but it takes a lot of time. There is a process that gives us the exact answer, which we will explore in the screen "Square roots."

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3. The Pythagorean theorem

Example 1: Find the missing side of the triangle you see when you click the icon.

As some of you may know, there is a formula that relates the three sides of a [right triangle](#) called the Pythagorean theorem. The formula is $a^2 + b^2 = c^2$, where c is the longest side of the [right triangle](#) (called the hypotenuse, and always opposite the 90° angle), and a and b are the other two sides of that triangle. Using it, we get:

$$3^2 + 4^2 = x^2$$
$$9 + 16 = 25 = x^2$$

So, what number squared equals 25? The answer is 5.

Example 2: Find the missing side of the triangle you see when you click the icon.

Again, we use the Pythagorean theorem, which yields:

$$5^2 + 5^2 = x^2$$
$$25 + 25 = 50 = x^2$$

So, what number squared equals 50? This time, the answer is not an integer. Instead, it will be a number between 7 and 8. This is because $7^2 = 49$ and $8^2 = 64$, and 50 is between 49 and 64. By guessing and checking, you could find that 7.07 is close, but that takes a fair amount of work, and you won't obtain an exact figure, since $7.07^2 = 49.9849$ —not 50. Again, there is a way to get the exact answer, which requires figuring out a method for undoing squaring.

The undoing of addition is subtraction.
The undoing of multiplication is division.
The undoing of squaring is... "square rooting."

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4. Square roots

Square roots arise in a variety of situations when we are trying to reverse the operation of squaring. If someone gives us a square that has an area of 25 square cm, it is pretty easy to guess and check and figure out that the sides of the square are 5 cm. However, when the area of a square is 23 square cm, guessing and checking does not seem to lead to an exact answer—we need to find the [square root](#).

The [radical](#) and the [radicand](#) make up the notation used for square roots. Click the icon to see this notation.

We now solve an equation in which the variable is squared by square rooting both sides. It is important to make sure the variable is alone on one side before you do this.

By square rooting both sides, we get the equation below.

As seen below, square rooting undoes squaring, and squaring undoes square rooting.

You might want to check that this is true on your calculator using the squaring key (x^2) and the [square root](#) key (the key with the [radical](#) sign shown in the figure above).

Finding the [square root](#) of a number like 23 used to be a difficult process, but now most calculators can find an excellent *approximation* to the [square root](#) of 23 in the blink of an eye. Our calculator says that it is 4.795831523. Other approximations are 4.796 and 5. They are numerical answers that are accurate to a certain decimal point. When a calculator shows a decimal stretching across the entire screen, it usually means that the decimal goes on forever.

Sure enough, the [square root](#) of 23 has an infinite, non-repeating decimal expression. Usually, truncating to 4.79, or rounding to 4.80, is good enough; but, as you proceed in math, there will be times when your teacher asks for the exact answer. At that time, you will have to write your answer in [radical](#) form, such as:

Much of this guide covers working with these strange kinds of numbers.

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5. Peculiarities of square roots and radical notation

The [radical](#) notation for the [square root](#) of 23 shown above represents the exact [square root](#) of 23. On your paper, if you write down this expression, and then you square it, you will get exactly 23, as shown below.

When you find the [square root](#) of 23 on your calculator by hitting the [square root](#) key (radical key), your calculator will display an approximation that looks something like this, 4.795831523. Then if you hit the squaring key (x^2), most calculators will display 23 again. While this is useful to show you that squaring the [square root](#) gets you back to 23, this can be somewhat misleading.

The decimal representation, 4.795831523, is only a very small part of the infinitely long decimal expression that makes up the exact [square root](#) of 23. The exact answer to $4.795831523 \times 4.795831523$ is 22.999999997000499529. Your calculator is usually rounding, or remembering stored information. For most people, this degree of precision is unimportant, however for scientists working with very large, or very small numbers, it is very important.

The only time a calculator can give you an exact answer when you ask it for a [square root](#) is when the number is a [perfect square](#). A [perfect square](#) is a number like 9, 16, or 25—numbers that can be obtained by squaring an integer.

A [radical](#) sign represents only the positive [square root](#)

The [radical](#) notation for the [square root](#) of 25, shown above, represents the positive [square root](#) of 25. The fact that $5^2 = 5 \cdot 5 = 25$, and $(-5)^2 = (-5) \cdot (-5) = 25$, indicates that all positive numbers have two square roots, a root that is negative, and a root that is positive. The number under the [radical](#) sign, shown above, is only equal to 5. If you want to show both square roots of 25, you must write:

Sometimes people write

Square roots of negative numbers don't exist (in the real number system).

You cannot find a real number multiplied by itself that results in a negative number. If the number is positive, its square will be positive. If the number is negative, its square will still be positive. For example, $(-5)(-5) = +25$. Therefore, if the [radicand](#) is negative, no real [square root](#) exists.

Eventually, you will learn about an imaginary number system that was created to work with such ideas!

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6. The need to reduce radicals and simple radical form

To find square roots, sometimes referred to as radicals, you perform a simplification process that is analogous to reducing a fraction; hence the expression "reducing radicals." This is part of a process called writing the [radical](#) in simple [radical](#) form. When you are reducing a fraction, you are rewriting it so that it represents the same number, but is in its most reduced form. There are many advantages to this. In particular, you can compare your fraction with another person's fraction and instantly tell if they are equal. For example, both fractions below are equal to $3/4$, yet they look like different numbers.

Miguel and April solved a problem involving radicals, and their answers are shown below. They looked up the answer in the back of the book, and it appeared to be different. The book answer is also shown below.

It turns out that all of these are the same number; and, in one sense, Miguel and April are both correct. If you calculate each number on your calculator correctly, you will get the same number, 8.485281374... (Make sure you can do this for each expression above!) However, at one point in your math classes, teachers will require that your answers be in simple [radical](#) form.

By comparing the decimal approximations of radicals, you can check many of your answers against problems on homework and quizzes. Sometimes it is nice to know that you have the correct answer even if you do not remember how to reduce radicals and get from

Practice using your calculator to confirm that Miguel's answer, April's answer, and the book answer, are all approximately equal to the decimal 8.485281374.

In the following two screens, you will learn two ways to reduce radicals. The first is an intuitive, guess-and-check method that works well in most situations. The second is a more formal approach that has different benefits.

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7. Reducing radicals by guess and check

We now show the process for reducing radicals. We will show an intuitive guess-and check-process on this screen and then a more formal approach on the next screen.

List the following perfect squares at the top of your paper: 4, 9, 16, 25, 36, 49, 64, 81, 100. Click the icon to see a table that you may find helpful.

Using your calculator or your own knowledge of factors, find the biggest [perfect square](#) that is a factor of the [radicand](#) (number inside the radical). As with reducing fractions, if you accidentally use a [perfect square](#) that is not the largest, that is not a problem, you can just keep reducing.

If you have a calculator, you can search for a [perfect square](#) factor by dividing 72 by each [perfect square](#) in your list. If the answer is a decimal, then it is not a factor. You never need to check perfect squares that are greater than half of the [radicand](#) (in this case, there would be no need to check perfect squares above 36). For the first [radical](#) above, the [radicand](#) is 72, and the biggest [perfect square](#) that is a factor of 72 is 36. Now, perform the following legal substitutions (rewriting of the expressions).

Check

The first substitution is obviously legal. The next substitution is legal because of one of the rules for radicals seen in the screen "Rules for radicals." The rule states that a [radical](#) of a product can be rewritten as the product of the radicals. The final substitution is obvious because the [square root](#) of 36 is 6.

We now look at the case where you accidentally do not pick the largest [perfect square](#). In this problem, you may have missed that 36 was a factor, but instead you noticed that 9 was a factor of 72. Then the substitution process would look like the following:

Notice that this is April's answer to the problem from the previous screen! You may feel that you have performed the reduction process, and you have. However, you must always check that your radicands are fully reduced (just like fractions must be fully reduced). In this case, the [radicand](#) 8 can be reduced, because the [perfect square](#) 4 is a factor of 8. The process would continue like this:

Notice that we end up with the same fully reduced [radical](#) expression. Sometimes, students write their steps without using the multiplication dot as we have shown above. They get to the following expression and ask, "What do I do with the 2 that I am going to get from the [square root](#) of 4?"

Even if you do not write the multiplication dot, it is critical to be aware that all relationships in this process have been multiplication. Again, remember to check on your calculator that your answer makes the same decimal expression as the original [radical](#).

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8. Reducing radicals by prime factorization

Sometimes you may be told to reduce radicals by first rewriting the [radicand](#) using its [prime factorization](#) (see the Britannica Study Guide *Integers: Dividing Integers and Factoring*). This approach is more algorithmic in that you will always get the complete reduction on the first try, and you do not need to be vigilant and check that all of your radicands are completely reduced. The same method (using prime factorization) is also used to reduce fractions—with similar advantages and drawbacks. Make strategic choices about which method works best for you. We show the [prime factorization](#) method below.

This method again makes use of the product rule of radicals. It also makes use of the fact that the [square root](#) of x times x is x , as shown below.

This result should be obviously true. Notice in the [prime factorization](#) method that we also used the fact that multiplication is commutative (you can multiply in any order). Remember to check your answer on your calculator.

You may finish by asking, "Why can't I just use my calculator to check whether two radicals are the same by seeing if they make the same decimal? Why do I have to reduce these radicals?" This is discussed in the next series of screens "Rules for radicals."

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9. Rules for radicals: avoiding temptations

What would you do if you were given the following expressions to simplify?

If you did not know otherwise, you could be tempted to do many things. You might think that the answer to A is the square root of 5, but it is not. You might be tempted to think that the answer to C is the square root of 6, but it is not. Right when you might be thinking that all of these are already simplified, we would tell you that A is the only expression that is fully simplified to simple radical form.

As you can see, these rules for radicals are not obvious! We'll begin by saying what you can't do, and showing you a method of checking and seeing for yourself what will work and what won't.

You can't ever perform a math operation between a radical, and a non-radical.

In other words,

All these answers seem as if they make sense, yet none are true. You can test your answers using your common sense. Replace all of the radicands with a perfect square whose root you know well, such as 16.

In other words, every time you are tempted to do an operation between a number and a radical, check by placing an obvious perfect square inside the radical to see for yourself whether you are allowed to perform the operation.

You are also not allowed to add or subtract radicals from each other. Again, we check with some perfect squares.

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10. Rules for radicals: what is allowed?

In the previous screen, we discussed that you should not be tempted to perform math operations between numbers and radicals. In this screen, we show you what operations are appropriate.

Product rule and quotient rule

The product rule is often applicable in the reverse direction. We have already shown this in earlier screens on reducing radicals. The basic rules say that when two radicals are multiplied or divided, you can put their product or quotient under one [radical](#) sign. Here are some examples:

Notice that we have used perfect squares, so that you can easily check to make sure that each side of the equation is true. 2 times 3 is 6, and 4 divided by 2 is 2!

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11. Rules for radicals: sometimes you can add!

In a previous screen, "Rules for radicals: avoiding temptations," we said that you cannot add radicals. This is true.

However, the situation is complicated by the fact that you can always add similar objects. After all, if you have 2 cases of CDs, and you add 3 cases of CDs, then you obviously have 5 cases of CDs. Similarly, if you have 2 of a certain radical, and 3 of the same radical, then you have 5 of that radical.

It is worth noting that this is always true, as seen in the following examples:

Example 1: Simplify the expression below.

In this case, simplify means "put in simple radical form." Although we cannot add like terms now, because the terms in their present form are unlike, after we reduce, we may generate like terms. The work is shown below:

As you can see, we were able to subtract these radicals after all, because as reduced radicals, they had equal radicands, and they were like terms. We practice this further in the following screen, "Simplifying to simple radical form."

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12. Simplifying to simple radical form

Radical expressions are in simple [radical](#) form when no [radicand](#) has a [perfect square](#) factor, no like terms exist, no radicals are in the denominators, and there are no parentheses.

Let's look at some examples.

Example 1: Simplify the expression below.

In this example, we can't work within the parenthesis, as these radicals are fully reduced, and the radicands are unequal. However, when we perform the distributive law, we have to reinvestigate whether the radicals can be reduced, as follows:

Notice that we used the [prime factorization](#) technique, to avoid having to multiply 6 by 15 to get a large number that would eventually be reduced in any case.

Example 2: Simplify the expression below.

It would appear that there is nothing to do to simplify this expression. However, if you look at the definition of simple [radical](#) form, it says that a [radical](#) may not be in a denominator. If you ever get this as an answer to a problem, it is unlikely that your answer will match the answer in the back of the book, because the answer in the back of the book will be in simple [radical](#) form. To get rid of the [radical](#) in the denominator, simply multiply the fraction by a version of 1, composed of the [radical](#) over itself, as follows:

Note that this technique will always work as long as you only have one term in the denominator. If you have more than one, you will have to use a [conjugate](#), which is covered in another study guide. When you get rid of the [radical](#) in the denominator, you [rationalize the denominator](#). After doing this, check that both expressions make the same decimal. In this case, the decimal should be 1.41421...

Example 3: Simplify the expression below.

You cannot cancel in this fraction because there is more than one term in the numerator (top). We begin by reducing the [radical](#).

You can further simplify by either factoring or "un-adding" the fraction. We show the latter:

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Glossary

conjugate

Given an expression $a + b$, the [conjugate](#) is $a - b$.

irrational numbers

numbers which have an infinite, non-repeating decimal expression. They cannot be expressed as a ratio of two integers. For example, the [square root](#) of 2 and π .

perfect square

a number which results from squaring an integer

prime factorization

expressing a number as a product of its prime factors

radical

symbol used to show the principal root of a quantity. For a [square root](#), this is the positive [square root](#).

radicand

expression under the [radical](#) sign

rationalize the denominator

to create an equivalent expression in which there is no [radical](#) in the denominator

right triangle

triangle with a 90° angle. The longest side, which is always opposite the 90° angle, is called the hypotenuse.

simple radical form

no radicands have [perfect square](#) factors, no like terms exist, no fractions have radicals in the denominators, and there are no parentheses

square root

given a number n , a [square root](#) of n is a number which makes n when multiplied by itself

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Teacher's Notes

The somewhat arcane rules for radicals are often taught in beginning algebra in a disembodied way, since the rest of the algebra curriculum does not use them very much. This study guide was designed for beginning algebra students who have moved through the coursework well enough to get to the work on roots and radicals, which is often at the end of an algebra textbook. We have tried to give them some context by beginning with discussions of finding missing sides of squares and right triangles using the Pythagorean theorem. Many schools and curriculums avoid these topics, by choice or by circumstance, until after the beginning algebra course. We expect that many geometry and advanced algebra students will benefit from the topics presented.

Geometry students are typically exposed to working with radicals first using the Pythagorean theorem, which we use to introduce the topic, and second using similar triangles, in particular working with 30-60-90° and 45° right triangles. Advanced algebra students initially need a review of simplifying radicals when they are exposed to working with the quadratic formula, and they definitely need one by the time they get to exponential models and logarithms. Even trigonometry students may benefit from a review as they learn the various exact values of the sine and cosine functions.

This guide maintains its introductory flavor by avoiding more difficult issues such as equations with radicals or complicated simplification problems. Nevertheless, we discuss some of the peculiarities of [radical](#) notation so that the student can be well prepared for the later subtleties of these topics.

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Help

How to use the Study Guides

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Navigation

The Contents list, which is located at the top of the module, includes all the topic headings in the particular Study Guide, as well as the Glossary, Teacher's Notes, and Help functions. All headings in the Contents list are hotlinked. If you click on one of the headings, you will reach the selected section of the document. Move back to the Contents list by clicking the "back to top" link, which is always located at the bottom of the particular section you are in, or move to the next heading following the one you are in, by clicking the second link, entitled "next - [content heading name]."

Links

The entire document can be scrolled and navigated sequentially like a Word document. External links and glossary terms are hotlinked as they occur throughout the document. Click on the relevant hotlinked URLs to be automatically connected to a live Web site that contains information relating to the Study Guide topic.

Glossary

Click on underlined words in the text to go to the Glossary for a definition. To see the entire glossary for this Study Guide, click on the Glossary link located at the end of the contents list.

Teacher's Notes

Click on the Teacher's Notes link located at the end of the Contents list at the top of the module to go to the Teacher's Notes section, which provides further lesson ideas and suggestions.

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